15

SETS, RELATIONS AND FUNCTIONS

Let us consider the following situation :

One day Mrs. and Mr. Mehta went to the market. Mr. Mehta purchased the following objects/items.

"a toy, one kg sweets and a magazine". Where as Mrs. Mehta purchased the following objects/items.

"Lady fingers, Potatoes and Tomatoes".

In both the examples, objects in each collection are well defined. What can you say about the collection of students who speak the truth ? Is it well defined? Perhaps not.

A set is a collection of well defined objects. For a collection to be a set it is necessary that it should be well defined.

The word well defined was used by the German Mathematician George Cantor (1845- 1918 A.D) to define a set. He is known as father of set theory. Now-a-days set theory has become basic to most of the concepts in Mathematics.

In our everyday life we come across different types of relations between the objects. The concept of relation has been developed in mathematical form.

The word function was introduced by Leibnitz in 1694. Function is a special type of relation. Each function is a relation but each ***relation is not a function***. In this lesson we shall discuss some basic definitions and operations involving sets, Cartesian product of two sets, relation between two sets, the conditions under when a relation becomes a function, different types of function and their properties.

OBJECTIVES

After studying this lesson, you will be able to :

- define a set and represent the same in different forms;
- define different types of sets such as, finite and infinite sets, empty set, singleton set, equivalent sets, equal sets, sub sets and cite examples thereof;
- define and cite examples of universal set, complement of a set and difference between two sets;
- define union and intersection of two sets;
- represent union and intersection of two sets, universal sets, complement of a set, difference between two sets by Venn Diagram;
- solve real life problems using Venn Diagram;
- define Cartesian product of two sets;
- define relation, function and cite examples thereof;
- find domain and range of a function;
- define and cite examples of diferent types of functions (one-one, many-one, onto, into and bijection);
- determine wheather a function is one-one, many-one, onto or into;
- draw the graph of functions;
- define and cite examples of odd and even functions;
- determine wheather a function is odd or even or neither;
- define and cite examples of functions like $|x|$, $[x]$ the greatest integer function, polynomial functions, logarithmic and exponential functions;
- define composition of two functions;
- define the inverse of a function; and
- state the conditions for the inverse to exist.

EXPECTED BACKGROUND KNOWLEDGE

- Number systems, concept of ordered pairs.

15.1 SOME STANDARD NOTATIONS

Before defining different terms of this lesson let us consider the following examples:

(i) collection of tall students in your school.	(i) collection of those students of your school whose height is more than 180 cm.
(ii) collection of honest persons in your colony.	(ii) collection of those people in your colony who have never been found involved in any theft case.
(iii) collection of interesting books in your school library.	(iii) collection of Mathematics books in your school library.
(iv) collection of intelligent students in your school.	(iv) collection of those students in your school who have secured more than 80% marks in annual examination.

In all collections written on left hand side of the vertical line the term tallness, interesting, honesty, intelligence are not well defined. In fact these notions vary from individual to individual. Hence those collections can not be considered as sets.

While in all collections written on right hand side of the vertical line, 'height' 'more than 180 cm.' 'mathematics books' 'never been found involved in theft case,' ' marks more than 80%' are well defined properties. Therefore, these collections can be considered as sets.

If a collection is a set then each object of this collection is said to be an element of this set. A set is usually denoted by capital letters of English alphabet and its elements are denoted by small letters.

For example, A = Toy elephant, packet of sweets, magazines.

Some standard notations to represent sets :

N : the set of natural numbers

W : the set of whole numbers

Z or I : the set of integers

Z^+ : the set of positve integers

Z^- : the set of negative integers

Q : the set of rational numbers

R : the set of real numbers

C : the set of complex numbers

Other frequently used symbols are :

$\in$: 'belongs to'

$\notin$: 'does not belong to'

$\exists$: There exists, $\nexists$: There does not exist.

For example N is the set of natural numbers and we know that 2 is a natural number but -2 is not a natural number. It can be written in the symbolic form as $2 \in N$ and $-2 \notin N$.

15. 2 REPRESENTATION OF A SET

There are two methods to represent a set.

15.2.1 (i) Roster method (Tabular form)

In this method a set is represented by listing all its elements, separating these by commas and enclosing these in curly bracket.

If V be the set of vowels of English alphabet, it can be written in Roster form as :

$V = \{ a, e, i, o, u\}$

(ii) If A be the set of natural numbers less than 7.

then $A=\{1, 2, 3, 4, 5, 6\}$, is in the Roster form.

Note : To write a set in Roster form elements are not to be repeated i.e. all elements are taken as distinct. For example if A be the set of letters used in the word mathematics, then

$A = \{m, a, t, h, e, i, c, s\}$

15.2.2 Set-builder form

In this form elements of the set are not listed but these are represented by some common property.

Let V be the set of vowels of English alphabet then V can be written in the set builder form as:

$V = \{x : x \text{ is a vowel of English alphabet}\}$

(ii) Let A be the set of natural numbers less than 7.

then $A = \{x : x \in N \text{ and } 1 \le x < 7\}$

Note : Symbol ':' read as 'such that'

Example: 15.1 Write the following in set -builder form :

(a) $A=\{-3,-2,-1,0,1,2,3\}$ (b) $B = \{3,6,9,12\}$

Solution : (a) $A = \{ x : x \in Z \text{ and } -3 \le x \le 3 \}$

(b) $B = \{ x : x = 3n \text{ and } n \in N, n \le 4 \}$

Example 15.2 Write the following in Roster form.

(a) $C = \{ x : x \in N \text{ and } 50 \le x \le 60 \}$

(b) $D = \left\{ x : x \in R \text{ and } x^2 - 5x + 6 = 0 \right\}$

Solution : (a) $C = \{50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60\}$

(b) $x^2 - 5x + 6 = 0$

$\Rightarrow$ $(x-3)(x-2) = 0$

$\Rightarrow \qquad x = 3, 2.$

$\therefore \qquad D = \{2, 3\}$

15.3 CLASSIFICATION OF SETS

15.3.1 Finite and infinite sets

Let A and B be two sets where

A = {x : x is a natural number}

B = {x : x is a student of your school}

As it is clear that the number of elements in set A is not finite (infinite) while number of elements in set B is finite. A is said to be an infinite set and B is said to be is finite set.

A set is said to be finite if its elements can be counted and it is said to be infinite if it is not possible to count upto its last element.

15.3.2 Empty (Null) Set : Consider the following sets.

$$A = \{x : x \in R \text{ and } x^2 + 1 = 0\}$$

B = {x : x is number which is greater than 7 and less than 5}

Set A consists of real numbers but there is no real number whose square is -1. Therefore this set consists of no element. Similiarly there is no such number which is less than 5 and greater than 7. Such a set is said to be a null (empty) set. It is denoted by the symbol void, ϕ or { }

A set which has no element is said to be a null/empty/void set, and is denoted by ϕ.

15.3.3 Singleton Set : Consider the following set :

A = {x : x is an even prime number}

As there is only one even prime number namely 2, so set A will have only one element. Such a set is said to be singleton. Here A = $\{2\}$.

A set which has only one element is known as singleton.

15.3.4 Equal and equivalent sets : Consider the following examples.

(i) $A = \{1, 2, 3\}$, $B = \{2, 1, 3\}$ (ii) $D = \{1, 2, 3\}$, $E = \{a, b, c\}$.

In example (i) Sets A and B have the same elements. Such sets are said to be equal sets and it is written as A = B. In example (ii) set D and E have the same number of elements but elements are different. Such sets are said to be equivalent sets and are written as A $\approx$ B.

Two sets A and B are said to be equivalent sets if they have same number of elements but they are said to be equal if they have not only the same number of elements but elements are also the same.

15.3.5 Disjoint Sets : Two sets are said to be disjoint if they do not have any common element. For example,sets A= { 1,3,5} and B = { 2,4,6 } are disjoint sets.

xample 15.3 Given that

$A = \{2, 4\}$ and $B = \{x : x$ is a solution of $x^2 + 6x + 8 = 0\}$

Are A and B disjoint sets ?

Solution : If we solve $x^2 + 6x + 8 = 0$, we get

$x = -4, -2$.

$\therefore$ $B = \{-4, -2\}$ and $A = \{2, 4\}$

Clearly ,A and B are disjoint sets as they do not have any common element.

Example 15.4 If $A = \{x : x$ is a vowel of English alphabet$\}$

and $B = \{y : y \in N \text{ and } y \le 5\}$

Is (i) $A = B$ (ii) $A \approx B$?

Solution : $A = \{a, e, i, o, u\}$, $b = \{1, 2, 3, 4, 5\}$.

Each set is having five elements but elements are different

$\therefore$ $A \neq B$ but $A \approx B$.

Example 15.5 Which of the following sets

$A = \{x : x$ is a point on a line$\}$

$B = \{y : y \in N$ and $y \le 50\}$

are finite or infinite ?

Solution : As the number of points on a line is uncountable (cannot be counted) so A is an infinite set while the number of natural numbers upto fifty can be counted so B is a finite set.

Example 15.6 Which of the following sets

$A = \{x : x \text{ is irrational and } x^2 - 1 = 0\}$.

$B = \{x : x \in z \text{ and } -2 \le x \le 2\}$ are empty?

Solution : Set A consists of those irrational numbers which satisfy $x^2 - 1 = 0$. If we solve $x^2 - 1 = 0$ we get $x = \pm 1$. Clearly ± 1 are not irrational numbers. Therefore A is an empty set. But $B = \{-2, -1, 0, 1, 2\}$. B is not an empty set as it has five elements.

Example 15.7 Which of the following sets are singleton ?

$A = \{x : x \in Z \text{ and } x - 2 = 0\}$ $\quad$ $B = \{y : y \in R \text{ and } y^2 - 2 = 0\}$.

Solution : Set A contains those integers which are the solution of $x - 2 = 0$ or $x = 2$. $\therefore$ $A = \{2\}$.

$\Rightarrow$ A is a singleton set.

B is a set of those real numbers which are solutions of $y^2 - 2 = 0$ or $y = \pm\sqrt{2}$

$\therefore$ $B = \{-\sqrt{2}, \sqrt{2}\}$ Thus, B is not a singleton set.

CHECK YOUR PROGRESS 15.1

1. Which of the following collections are sets ?
   (i) The collection of days in a week starting with S.
   (ii) The collection of natural numbers upto fifty.
   (iii) The collection of poems written by Tulsidas.
   (iv) The collection of fat students of your school.

(2) Insert the appropriate symbol in blank spaces.

If $A = \{1, 2, 3\}$.

(i) 1...........A (ii) 4........A.

3. Write each of the following sets in the Roster form :
   (i) $A = \{x : x \in z \text{ and } -5 \le x \le 0\}$.
   (ii) $B = \{x : x \in R \text{ and } x^2 - 1 = 0\}$.
   (iii) C = {x : x is a letter of the word banana}.
   (iv) D = {x : x is a prime number and exact divisor of 60}.

4. Write each of the following sets are in the set builder form ?
   (i) A = {2, 4, 6, 8, 10} (ii) B = {3, 6, 9,...... ∞ }
   (iii) C = {2, 3, 5, 7} (iv) $D = \{-\sqrt{2}, \sqrt{2}\}$

   Are A and B disjoints sets ?

5. Which of the following sets are finite and which are infinite ?
   (i) Set of lines which are parallel to a given line.
   (ii) Set of animals on the earth.
   (iii) Set of Natural numbers less than or equal to fifty.
   (iv) Set of points on a circle.

6. Which of the following are null set or singleton ?
   (i) $A = \{x : x \in R \text{ and } x \text{ is a solution of } x^2 + 2 = 0\}$.
   (ii) $B = \{x : x \in Z \text{ and } x \text{ is a solution of } x - 3 = 0\}$.
   (iii) $C = \{x : x \in Z \text{ and } x \text{ is a solution of } x^2 - 2 = 0\}$.
   (iv) D = {x : x is a student of your school studying in both the classes XI and XII }

7. In the following check whether A = B or $A \approx B$.
   (i) A = {a}, B = {x : x is an even prime number}.
   (ii) A = {1, 2, 3, 4}, B = {x : x is a letter of the word guava}.
   (iii) A = {x : x is a solution of $x^2 - 5x + 6 = 0$ },B = {2 , 3}.

15.4 SUB- SET

Let set A be a set containing all students of your school and B be a set containing all students of class XII of the school. In this example each element of set B is also an element of set A. Such a set B is said to be subset of the set A. It is written as $B \subseteq A$

Consider $D = \{1, 2, 3, 4, ...\}$

$E = \{.....-3-2,-1, 0, 1, 2, 3, ...\}$

Clearly each element of set D is an element of set E also $\therefore\ D \subseteq E$

If A and B are any two sets such that each element of the set A is an element of the set B also, then A is said to be a subset of B.

Remarks

(i) Each set is a subset of itself i.e. $A \subseteq A$.

(ii) Null set has no element so the condition of becoming a subset is automatically satisfied. Therefore null set is a subset of every set.

(iii) If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iv) If $A \subseteq B$ and $A \neq B$ then A is said to be a proper subset of B and B is said to be a super set of A. i.e. $A \subset B$ or $B \supset A$.

Example 15.8 If A = {x : x is a prime number less than 5} and

B = {y : y is an even prime number} then is B a proper subset of A ?

Solution : It is given that

$A = \{2, 3\}, \quad B = \{2\}$.

Clearly $B \subseteq A$ and $B \neq A$

We write $B \subset A$

and say that B is a proper subset of A.

Example 15.9 If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$.

is $A \subseteq B$ or $B \subseteq A$?

Solution : Here $1 \in A$ but $1 \notin B \Rightarrow A \not\subseteq B$.

Also $5 \in B$ but $5 \notin A \Rightarrow B \not\subseteq A$.

Hence neither A is a subset of B nor B is a subset of A.

Example 15.10 If A = {a, e, i, o, u}

B = {e, i, o, u, a }

Is $A \subseteq B$ or $B \subseteq A$ or both ?

Solution : Here in the given sets each element of set A is an element of set B also

$\therefore \quad A \subseteq B$ .... (i)

and each element of set B is an element of set A also. $\therefore\ B \subseteq A$ ......(ii)

From (i) and (ii)

$A = B$

15.5 POWER SET

Let $A = \{a, b\}$

Subset of A are ϕ, {a}, {b} and {a, b}.

If we consider these subsets as elements of a new set B (say) then

$B = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

B is said to be the power set of A.

Notation : Power set of a set A is denoted by P(A).
Power set of a set A is the set of all subsets of the given set.

Example 15.11 Write the power set of each of the following sets :

(i) $A = \left\{x : x \in R \text{ and } x^2 + 7 = 0\right\}$.

(ii) $B = \left\{y : y \in N \text{ and} 1 \le y \le 3\right\}$.

Solution :

(i) Clearly $A = \phi$ (Null set)

$\therefore$ ϕ is the only subset of given set

$\therefore \quad P(A) = \{\phi\}$

(ii) The set B can be written as {1, 2, 3}

Subsets of B are ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}.

$\therefore \quad P(B) = \{\ \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\ \}$.

15.6 UNIVERSAL SET

Consider the following sets.

A = {x : x is a student of your school}
B = {y : y is a male student of your school}
C = {z : z is a female student of your school}
D = {a : a is a student of class XII in your school}

Clearly the set B, C, D are all subsets of A.

A can be considered as the universal set for this particular example. Universal set is generally denoted by U.

In a particular problem a set U is said to be a universal set if all the sets in that problem are subsets of U.

Remarks

(i) Universal set does not mean a set containing all objects of the universe.

(ii) A set which is a universal set for one problem may not be a universal set for another problem.

Example 15.12 Which of the following set can be considered as a universal set ?

X = {x : x is a real number}

Y = {y : y is a negative integer}

Z = {z : z is a natural number}

Solution : As it is clear that both sets **Y** and **Z** are subset of **X**.

∴ **X** is the universal set for this problem.

15.7 VENN DIAGRAM

British mathematician John Venn (1834 – 1883 AD) introduced the concept of diagrams to represent sets. According to him universal set is represented by the interior of a rectangle and other sets are represented by interior of circles.

For example if U= {1, 2, 3, 4, 5}, A = {2, 4} and B = {1,3}, then these sets can be represented as

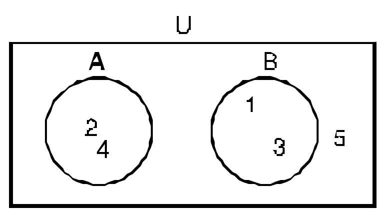


Fig. 15.1

Diagramatical representation of sets is known as a Venn diagram.

15.8 DIFFERENCE OF SETS

Consider the sets

A = {1, 2, 3, 4, 5} and B= {2, 4, 6}.

A new set having those elements which are in A but not B is said to be the difference of sets A and B and it is denoted by A – B.

∴ A – B= {1, 3, 5}

Similiarly a set of those elements which are in B but not in A is said to be the difference of B and A and it is devoted by B – A.

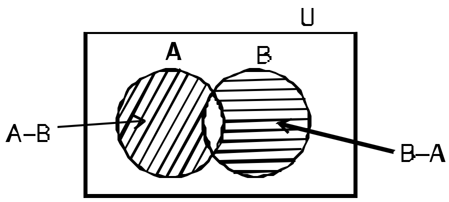
∴ B – A = {6}

In general, if A and B are two sets then

$$A - B = \{ x : x \in A \text{ and } x \notin B \}$$

$$B - A = \{ x : x \in B \text{ and } x \notin A \}$$

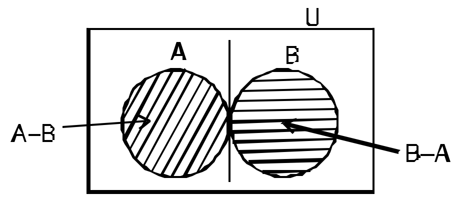
Difference of two sets can be represented using Venn diagram as :



When A and B are not **disjoint** sets

Fig. 15.2

or



When A and B are **disjoint** sets

Fig. 15.3

15.9. COMPLEMENT OF A SET

Let **X** denote the universal set and **Y, Z** its sub set where

X = {x : x is any member of the family}

Y = {x : x is a male member of the family}

Z = {x : x is a female member of the family}

$X - Y$ is a set having female members of the family.

$X - Z$ is a set having male members of the family.

$X - Y$ is said to be the complement of Y and is usally denoted by Y' or Y^c.

$X - Z$ is said to be complement of Z and denoted by Z' or Z^c.

If U is the universal set and A is its subset then the complement of A is a set of those elements which are in U which are not in A. It is denoted by A' or A^c.

$$A' = U - A = \{x : x \in U \text{ and } x \notin A\}$$

The complement of a set can be represented using Venn diagram as :

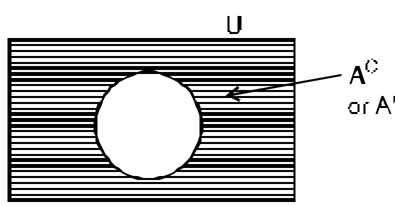


Fig. 15.4

Remarks

(i) Difference of two sets can be found even if none is a subset of the other but complement of a set can be found only when the set is a subset of some universal set.

(ii) $\phi^c = U$. (iii) $U^c = \phi$.

Example 15.13 Given that

A = {x : x is a even natural number less than or equal to 10}

and B = {x : x is an odd natural number less than or equal to 10}

Find (i) $A-B$ (ii) $B-C$ (iii) is $A-B=B-A$ **?**

Solution : It is given that

A = {2, 4, 6, 8, 10}, B = {1, 3, 5, 7, 9}

Therefore,

(i) $A-B$ ={2, 4, 6, 8, 10}

(ii) $B-A$ ={1, 3, 5, 7, 9}

(iii) Clearly from (i) and (ii) $A-B \neq B-A$.

Example 15.14 Let U be the universal set and A its subset where

U={ x : x $\in$ N and $x \leq 10$ }

A = {y : y is a prime number less than 10}

Find (i) A^c (ii) Represent A^c in Venn diagram.

Solution : It is given

U= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

A = { 2, 3, 5, 7}

(i) $A^c = U - A$ = {1, 4, 6, 8, 9, 10}

(ii)

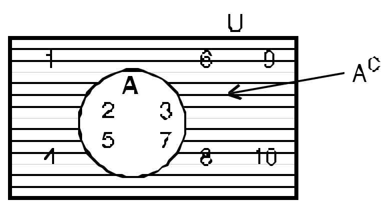


Fig. 15.5

CHECK YOUR PROGRESS 15.2

1. Insert the appropriate symbol in the blank spaces, given that A={1, 3, 5, 7, 9}

   (i) ϕ ....A (ii) {2, 3, 9}..........A

   (iii) 3............A (iv) 10...................A

2. Given that A = {a, b}, how many elements P(A) has ?

3. Let A = {ϕ, {1} ,{2}, {1,2}}

   Which of the following is true or false ?

Notes

(i) $\{1,2\} \subset A$ (ii) $\phi \in A$.

4. Which of the following statements are true or false ?
   (i) Set of all boys, is contained in the set of all students of your school.
   (ii) Set of all boy students of your school, is contained in the set of all students of your school.
   (iii) Set of all rectangles, is contained in the set of all quadrilaterals.
   (iv) Set of all circles having centre at origin is contained in the set of all ellipses having centre at origin.
5. If $A = \{1, 2, 3, 4, 5\}$, $B = \{5, 6, 7\}$ find (i) $A - B$ (ii) $B - A$.
6. Let N be the universal set and A, B, C, D be its subsets given by

   $A = \{x : x$ is a even natural number$\}$

   $B = \{x : x \in N$ and x is a multiple of 3$\}$

   $C = \{x : x \in N$ and $x \geq 5\}$

   $D = \{x : x \in N$ and $x \leq 10\}$

   Find complements of A, B, C and D respectively.

15.10. INTERSECTION OF SETS

Consider the sets

$A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$

It is clear, that there are some elements which are common to both the sets A and B. Set of these common elements is said to be interesection of A and B and is denoted by $A \cap B$.

Here $A \cap B = \{2, 4\}$

If A and B are two sets then the set of those elements which belong to both the sets is said to be the intersection of A and B. It is devoted by $A \cap B$.

$A \cap B = \{x : x \in A$ and $x \in B\}$

$A \cap B$ can be represented using Venn diagram as :

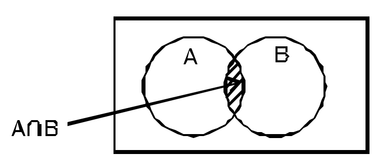


Fig. 15.6

Remarks

If $A \cap B = \phi$ then A and B are said to be **disjoint sets**. In Venn diagram disjoint sets can be represented as

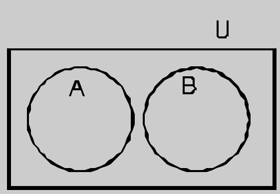


Fig.15.7

Example 15.15 Given that

$$A = \{x : x \text{ is a king out of 52 playing cards}\}$$

and $$B = \{ y : y \text{ is a spade out of 52 playing cards}\}$$

Find (i) $A \cap B$ (ii) Represent $A \cap B$ by using Venn diagram .

Solution : (i) As there are only four kings out of 52 playing cards, therefore the set A has only four elements. The set B has 13 elements as there are 13 spade cards but out of these 13 spade cards there is one king also. Therefore there is one common element in A and B. $\therefore A \cap B =$ { King of spade}.

(ii)

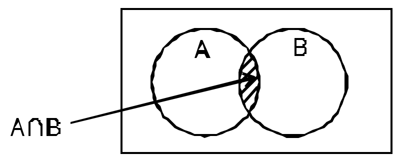


Fig. 15.8

15.11 UNION OF SETS

Consider the following examples :

(i) A is a set having all players of Indian men cricket team and B is a set having all players of Indian women cricket team. Clearly A and B are disjoint sets. Union of these two sets is a set having all players of both teams and it is denoted by $A \cup B$.

(ii) D is a set having all players of cricket team and E is the set having all players of Hockey team, of your school. Suppose three players are common to both the teams then union of D and E is a set of all players of both the teams but three common players to be written once only.

If A and B are only two sets then union of A and B is the set of those elements which belong to A or B.

In set builder form :

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

OR

$$A \cup B = \{x : x \in A - B \text{ or } x \in B - A \text{ or } x \in A \cap B\}$$

$A \cup B$ can be represented using Venn diagram as :

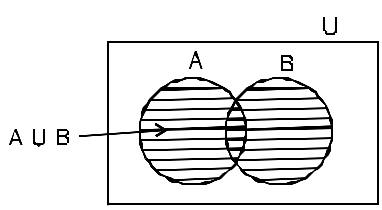


Fig. 15.9

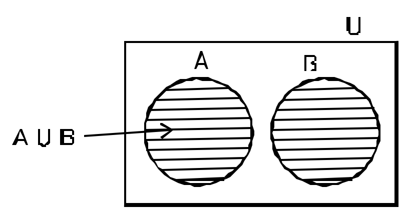


Fig. 15.10

$$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B).$$

or $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

where n $A \cup B$ stands for number of elements in $A \cup B$ so on.

Example 15.16 $\quad A = \{x : x \in Z^{+} \text{ and } \leq 5\}$

$B = \{y : y$ is a prime number less than $10\}$

Find (1) $A \cup B$ (ii) represent $A \cup B$ using Venn diagram.

Solution : We have,

A={1, 2, 3, 4, 5} B = {2, 3, 5, 7}.

$\therefore \quad A \cup B = \{1, 2, 3, 4, 5, 7\}$.

(ii)

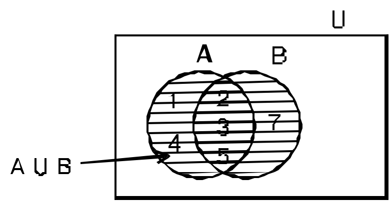


Fig.15.11

CHECK YOUR PROGRESS 15.3

1. Which of the following pairs of sets are disjoint and which are not ?

   (i) {x : x is an even natural number}, {y : y is an odd natural number}

   (ii) {x : x is a prime number and divisor of 12}, $\{y : y \in N \text{ and } 3 \leq y \leq 5\}$

   (iii) {x :x is a king of 52 playing cards}, { y : y is a diamond of 52 playing cards}

   (iv) {1, 2, 3, 4, 5}, {a, e, i, o, u}

2. Find the intersection of A and B in each of the following :

   (i) $A = \{x : x \in Z\}$, $B = \{x : x \in N\}$ (ii) A = {Ram, Rahim, Govind, Gautam}

   B = {Sita, Meera, Fatima, Manprit}

3. Given that A = {1, 2, 3, 4, 5}, B={5, 6, 7, 8, 9, 10}

   find (i) $A \cup B$ (ii) $A \cap B$.

4. If $A = \{x : x \in N\}$, $B = \{y : y \in z \text{ and } -10 \leq y \leq 0\}$ find $A \cup B$ and write your answer in the Roster form as well as set-builder form.

5. If A ={2, 4, 6, 8, 10}, B {8, 10, 12, 14}, C ={14, 16, 18, 20}.

   Find (i) $A \cup (B \cup C)$ (ii) $A \cap (B \cap C)$.

6. Let U = {1, 2, 3,.......10}, A = {2, 4, 6, 8, 10}, B = { 1, 3, 5, 7, 9, 10}

   Find (i) $(A \cup B)'$ (ii) $(A \cap B)'$ (iii) (B')' (iv) $(B - A)'$.

7. Draw Venn diagram for each the following :

(i) $A \cap B$ when $B \subset A$ (ii) $A \cap B$ when A and B are disjoint sets.

(iii) $A \cap B$ when A and B are neither subsets of each other nor disjoint sets.

8. Draw Venn diagram for each the following :

(i) $A \cup B$ when $A \subset B$. (ii) $A \cup B$ when A and B are disjoint sets.

(iii) $A \cup B$ when A and B are neither subsets of each other nor disjoint sets.

9. Draw Venn diagram for each the following :

(i) $A - B$ and $B - A$ when $A \subset B$.

(ii) $A - B$ and $B - A$ when A and B are disjoint sets.

(iii) $A - B$ and $B - A$ when A and B are neither subsets of each other nor disjoint sets.

15.12 CARTESIAN PRODUCT OF TWO SETS

Consider two sets A and B where

$A=\{1, 2\}$, $B= \{3, 4, 5\}$.

Set of all ordered pairs of elements of A and B

is $\{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

This set is denoted by $A \times B$ and is called the cartesian product of sets A and B.

i.e. $A\times B =\{(1, 3), (1, 4),(1, 5),(2, 3),(2, 4),(2, 5)\}$

Cartesian product of B sets and A is denoted by $B\times A$.

In the present example, it is given by

$B\times A = \{(3, 1),(3, 2),(4, 1),(4, 2),(5, 1),(5, 2)\}$

Clearly $A\times B \neq B\times A$.

In the set builder form :

$A\times B = \{(a,b) : a\in A \text{ and } b\in B\}$

$B\times A = \{(b,a) : b\in B \text{ and } a\in A\}$

Note : If $A = \phi$ or $B = \phi$ or $A, B = \phi$

then $A \times B = B \times A = \phi$.

Example 15.17

(1) Let $A=\{a,b,c\}$, $B=\{d,e\}$, $C=\{a,d\}$.

Find (i) $A\times B$ (ii) $B\times A$ (iii) $A\times(B \cup C)$ (iv) $(A \cap C)\times B$

(v) $(A \cap B)\times C$ (vi) $A \times (B - C)$.

Solution : (i) $A\times B =\{(a, d),(a, e), (b, d), (b, e), (c, d), (c, e)\}$.

(ii) $B\times A = \{(d, a),(d, b), (d, c), (e, a) (e, b),(e, c)\}$.

(iii) $A = \{a, b, c\}$, $B \cup C =\{a,d,e\}$.

$\therefore$ $A \times (B \cup C)$ ={(a, a),(a, d),(a, e),(b, a),(b, d),(b, e), (c, a),(c, d),(c, e).

(iv) $A \cap C = \{a\}$, B={d, e}.

$\therefore$ $(A \cap C) \times B$={(a, d), (a, e)}

(v) $A \cap B = \phi$, c={a,d}, $\therefore$ $A \cap B \times c = \phi$

(vi) A = {a,b,c}, $B - C = \{e\}$.

$\therefore$ $A \times (B - C) = \{(a,e),(b,e),(c,e)\}$.

15.13 RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim}

B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly $R \subseteq A \times B$, i.e.$R = \{(a,b): a \in A, b \in B$ and aRb}

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) $R = \phi$, R is called a void relation.

(ii) R=A×B, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) $R = \{ (a,a)\ \forall a \in A \}$, is called the identity relation.

15.13.1 Domain and Range of a Relation

If R is a relation between two sets then the set of its first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation.

Consider previous example given above.

Domain = {Mohan, Sohan, David, Karim}

Range = {Rita, Marry, Fatima}

Example 15.18 Given that A = {2, 4, 5, 6, 7}, B = {2, 3}.

R is a relation from A to B defined by

$R = \{(a, b) : a \in A, \ b \in B$ and a is divisible by b$\}$

find (i) R in the roster form (ii) Domain of R (iii) Range of R

(iv) Repersent R diagramatically.

Solution : (i) R = {(2, 2), (4, 2), (6, 2), (6, 3)}

(ii) Domain of R = {2, 4, 6}

(iii) Range of R = {2, 3}

(iv)

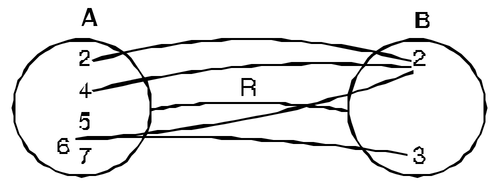


Fig. 15.12

Example 15.19 If R is a relation 'is greater than' from A to B, where A= {1, 2, 3, 4, 5} and B = {1,2,6}.

Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) R = {(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(ii) Domain of R = {3, 4, 5}

(iii) Range of R = {1, 2}

CHECK YOUR PROGRESS 15.4

1. Given that A = {4, 5, 6, 7}, B = {8, 9}, C = {10}

   Verify that $A \times (B - C) = (A \times B) - (A \times C)$.

2. If U is a universal set and A, B are its subsets.

   Where U= {1, 2, 3, 4, 5}.

   A = {1,3,5}, B = {x : x is a prime number} find A' × B'

3. If A = {4, 6, 8, 10}, B = {2, 3, 4, 5}

   R is a relation defined from A to B where

   $R = \{(a, b) : a \in A, \ b \in B$ and a is a multiple of b$\}$

   find (i)R in the Roster form (ii) Domain of R (iii) Range of R .

4. If R be a relation from N to N defined by

   $R = \{(x,y) : 4x + y = 12, \ x, y \in N\}$

   find (i) R in the Roster form (ii) Domain of R (iii) Range of R.

5. If R be a relation on N defined by

   $R = \{(x, x^2) : x$ is a prime number less than 15$\}$

Find (i) R in the Roster form (ii) Domain of R (iii) Range of R

6. If R be a relation on set of real numbers defined by R={(x,y) : $x^2 + y^2 = 0$} Find

(i) R in the Roster form (ii) Domain of R (iii) Range of R.

15.14 DEFINITION OF A FUNCTION

Consider the relation

f : {(a, 1), (b, 2), (c, 3), (d, 5)}

In this relation we see that each element of A has a unique image in *B*

This relation *f* from set *A* to *B* where every element of *A* has a unique image in *B* is defined as a function from *A* to *B*. So we observe that ***in a function no two ordered pairs have the same first element.***

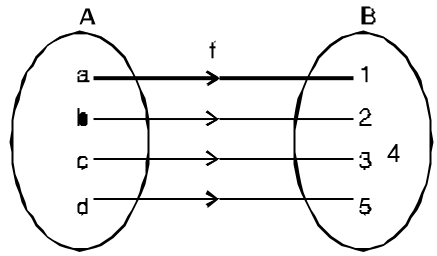


Fig.15.13

We also see that $\exists$ an element $\in$ B, i.e., 4 which does not have its preimage in A. Thus here:

(i) the set B will be termed as co-domain and

(ii) the set {1, 2, 3, 5} is called the range.

From the above we can conclude that ***range is a subset of co-domain.***

Symbolically, this function can be written as

$f : A \to B$ or $A \xrightarrow{f} B$

Example 15.20 Which of the following relations are functions from A to B. Write their domain and range. If it is not a function give reason ?

(a) $\{(1,-2),(3,7),(4,-6),(8,1)\}$, $A = \{1,3,4,8\}$, $B = \{-2,7,-6,1,2\}$

(b) $\{(1,0),(1-1),(2,3),(4,10)\}$, $A = \{1,2,4\}$, $B = \{0,-1,3,10\}$

(c) $\{(a,b),(b,c),(c,b),(d,c)\}$, $A = \{a,b,c,d,e\}$ $B = \{b,c\}$

(d) $\{(2,4),(3,9),(4,16),(5,25),(6,36\}$, $A = \{2,3,4,5,6\}$, $B = \{4,9,16,25,36\}$

(e) $\{(1,-1),(2,-2),(3,-3),(4,-4),(5,-5)\}$, $A = \{0,1,2,3,4,5\}$,

$B = \{-1,-2,-3,-4,-5\}$

(f) $\left\{\left(\sin\frac{\pi}{6},\frac{1}{2}\right),\left(\cos\frac{\pi}{6},\frac{\sqrt{3}}{2}\right),\left(\tan\frac{\pi}{6},\frac{1}{\sqrt{3}}\right),\left(\cot\frac{\pi}{6},\sqrt{3}\right)\right\}$,

$$A = \left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\} \quad B = \left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3},1\right\}$$

(g) $\{(a,b),(a,2),(b,3),(b,4)\}$, $A = \{a,b\}$, $B = \{b,2,3,4\}$.

Solution :

(a) It is a function.

Domain $= \{1,3,4,8\}$, Range $= \{-2,7,-6,1\}$

(b) It is not a function. Because Ist two ordered pairs have same first elements.

(c) It is not a function.

Domain $= \{a,b,c,d\} \neq A$, Range $= \{b,c\}$

(d) It is a function.

Domain $= \{2,3,4,5,6\}$, Range $= \{4,9,16,25,36\}$

(e) It is not a function .

Domain $= \{1,2,3,4,5\} \neq A$, Range $= \{-1,-2,-3,-4,-5\}$

(f) It is a function .

Domain $= \left\{\sin\frac{\pi}{6}, \cos\frac{\pi}{6}, \tan\frac{\pi}{6}, \cot\frac{\pi}{6}\right\}$, Range $= \left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}\right\}$

(g) It is not a function.

First two ordered pairs have same first component and last two ordered pairs have also same first component.

Example 15.21 State whether each of the following relations represent a function or not.

(a)

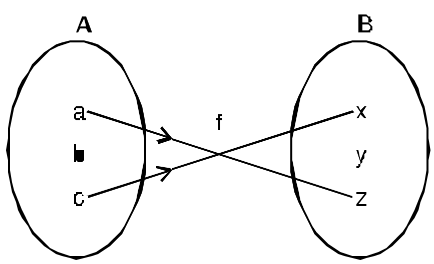


Fig.15.14

(b)

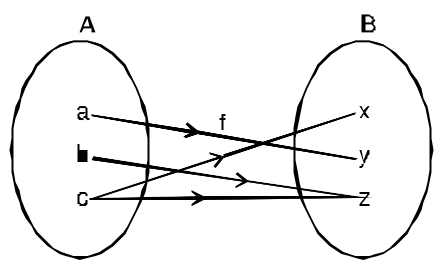


Fig.15.15

(c)

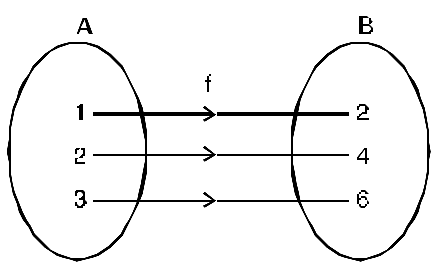


Fig. 15.16

(d)

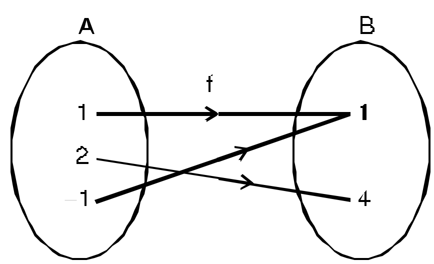


Fig.15.17

Solution :

(a) f is not a function because the element b of A does not have an image in B.

(b) f is not a function because the element c of A does not have a unique image in B.

(c) f is a function because every element of A has a unique image in B.

(d) f is a function because every element in A has a unique image in B.

Example 15.22 Which of the following relations from $R \rightarrow$. R are functions?

(a) $y = 3x + 2$ (b) $y < x + 3$ (c) $y = 2x^2 + 1$

Solution :

(a) $y = 3x + 2$

Here corresponding to every element $x \in R, \exists$ a unique element $y \in R$.

$\therefore$ It is a function.

(b) $y < x + 3$.

For any real value of x we get more than one real value of y.

$\therefore$ It is not a function.

(c) $y = 2x^2 + 1$

For any real value of x, we will get a unique real value of y.

$\therefore$ It is a function.

CHECK YOUR PROGRESS 15.5

1. Which of the following relations are functions from A to B ?

(a) $\{(1,-2),(3,7),(4,-6),(8,11)\}$, $A = \{1,3,4,8\}$, $B = \{-2,7,-6,11\}$

(b) $\{(1,0),(1,-1),(2,3),(4,10)\}$, $A = \{1,2,4\}, B = \{1,0,-1,3,10\}$

(c) $\{(a,2),(b,3),(c,2),(d,3)\}$, $A = \{a,b,c,d\}, B = \{2,3\}$

(d) $\{(1,1),(1,2),(2,3),(-3,4)\}$, $A = \{1,2,-3\}, B = \{1,2,3,4\}$

(e) $\left\{\left(2,\frac{1}{2}\right),\left(3,\frac{1}{3}\right),\ldots,\left(10,\frac{1}{10}\right)\right\}$,

$$A = \{1,2,3,4\}, B = \left\{\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{11}\right\}$$

(f) $\{(1,1),(-1,1),(2,4),(-2,4)\}$, $A = \{0,1,-1,2,-2\}, B = \{1,4\}$

2. Which of the following relations represent a function ?

(a) (b)

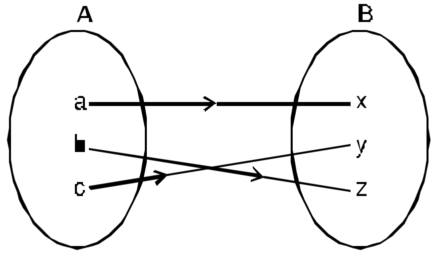


Fig. 15.18

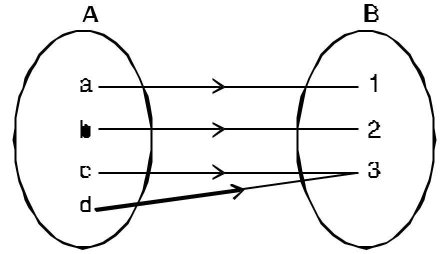


Fig. 15.19

(c) (d)

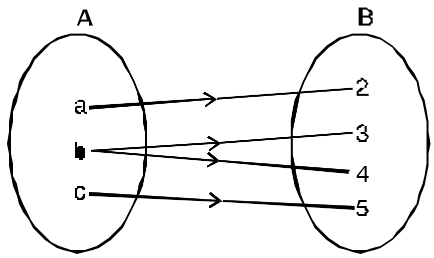


Fig. 15.20

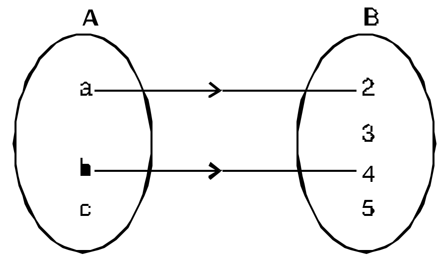


Fig. 15.21

3. Which of the following relations defined from $R \to R$ are functions ?

(a) $y = 2x + 1$ (b) $y > x + 3$ (c) $y < 3x + 1$ (d) $y = x^2 + 1$

4. Write domain and range for each of the following functions :

(a) $\left\{\left(\sqrt{2}, 2\right), \left(\sqrt{5}, -1\right), \left(\sqrt{3}, 5\right)\right\}$

(b) $\left\{\left(-3, \frac{1}{2}\right), \left(-2, \frac{1}{2}\right), \left(-1, \frac{1}{2}\right)\right\}$

(c) $\{(1,1), (0,0), (2,2), (-1,-1)\}$

(d) $\{(\text{Deepak}, 16), (\text{Sandeep}, 28), (\text{Rajan}, 24)\}$

5. Write domain and range for each of the following mappings :

(a) (b)

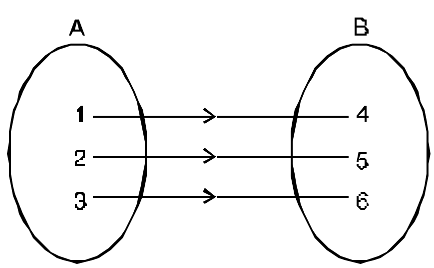


Fig. 15.22

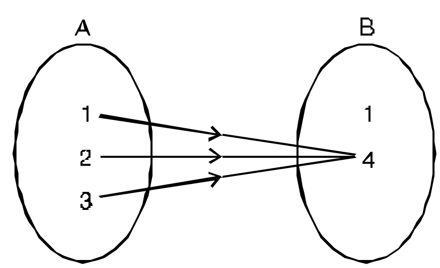
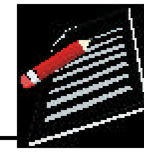


Fig. 15.23



Notes

(c)

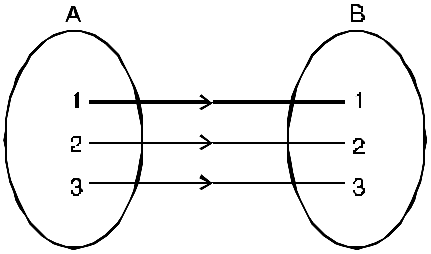


Fig. 15.24

(d)

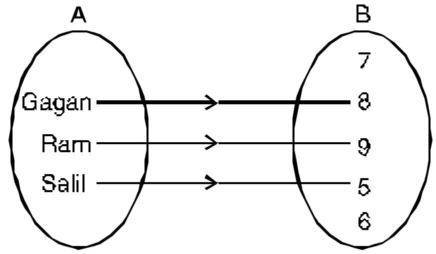


Fig. 15.25

(e)

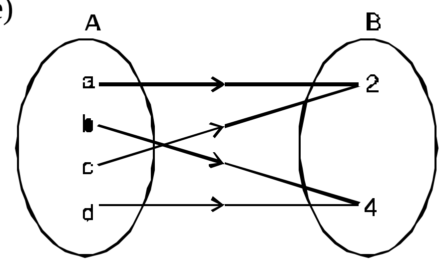


Fig. 15.26

15.14.1 Some More Examples on Domain and Range

Let us consider some functions which are only defined for a certain subset of the set of real numbers.

Example 15.23 Find the domain of each of the following functions :

(a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x-2}$ (c) $y = \frac{1}{(x+2)(x-3)}$

Solution : The function $y = \frac{1}{x}$ can be described by the following set of ordered pairs.

$$\left\{ ....,\left(-2,-\frac{1}{2}\right),(-1,-1),(1,1)\left(2,\frac{1}{2}\right),....\right\}$$

Here we can see that x can take all real values except 0 because the corresponding image, i.e., $\frac{1}{0}$ is not defined.

$\therefore$ Domain $= R - \{0\}$ [Set of all real numbers except 0]

Note : Here range = R - { 0 }

(b) x can take all real values except 2 because the corresponding image, i.e., $\frac{1}{(2-2)}$ does not exist.

$\therefore$ Domain $= R - \{2\}$

(c) Value of y does not exist for $x = -2$ and $x = 3$

$\therefore$ Domain $= R - \{-2, 3\}$

Example 15.24 Find domain of each of the following functions :

(a) $y = +\sqrt{x-2}$ (b) $y = +\sqrt{(2-x)(4+x)}$

Solution :(a) Consider the function $y = +\sqrt{x-2}$

In order to have real values of y, we must have $(x-2) \geq 0$

i.e. $x \geq 2$

$\therefore$ Domain of the function will be all real numbers ≥ 2.

(b) $y = +\sqrt{(2-x)(4+x)}$

In order to have real values of y, we must have $(2-x)(4+x) \geq 0$

We can achieve this in the following two cases :

Case I : $(2-x) \geq 0$ and $(4+x) \geq 0$

$\Rightarrow$ $x \leq 2$ and $x \geq -4$

$\therefore$ Domain consists of all real values of x such that $-4 \leq x \leq 2$

Case II : $2-x \leq 0$ and $4+x \leq 0$

$\Rightarrow$ $2 \leq x$ and $x \leq -4$.

But, x cannot take any real value which is greater than or equal to 2 and less than or equal to -4.

$\therefore$ From both the cases, we have

Domain $= -4 \leq x \leq 2 \ \forall \ x \in R$

Example 15.25 For the function

$f(x) = y = 2x + 1,$ find the range when domain $= \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution : For the given values of *x*, we have

$f(-3) = 2(-3) + 1 = -5$

$f(-2) = 2(-2) + 1 = -3$

$f(-1) = 2(-1) + 1 = -1$

$f(0) = 2(0) + 1 = 1$

$f(1) = 2(1) + 1 = 3$

$f(2) = 2(2) + 1 = 5$

$f(3) = 2(3) + 1 = 7$

The given function can also be written as a set of ordered pairs.

i.e., $\{(-3,-5),(-2,-3),(-1,-1),(0,1)(1,3),(2,5)(3,7)\}$

$\therefore$ Range $= \{-5,-3,-1,1,3,5,7\}$

Example 15.26 If $f(x) = x+3,\ 0 \le x \le 4$, find its range.

Solution : Here $0 \le x \le 4$

or $0+3 \le x+3 \le 4+3$

or $3 \le f(x) \le 7$

$\therefore$ Range $= \{f(x) : 3 \le f(x) \le 7\}$

Example 15. 27 If $f(x) = x^2,\ -3 \le x \le 3$, find its range.

Solution : Given $-3 \le x \le 3$

or $0 \le x^2 \le 9$ or $0 \le f(x) \le 9$

$\therefore$ Range $= \{f(x) : 0 \le f(x) \le 9\}$

CHECK YOUR PROGRESS 15.6

1. Find the domain of each of the following functions $x \in R$:

(a) (i) $y = 2x$ (ii) $y = 9x+3$ (iii) $y = x^2+5$

(b) (i) $y = \dfrac{1}{3x-1}$ (ii) $y = \dfrac{1}{(4x+1)(x-5)}$

(iii) $y = \dfrac{1}{(x-3)(x-5)}$ (iv) $y = \dfrac{1}{(3-x)(x-5)}$

(c) (i) $y = \sqrt{6-x}$ (ii) $y = \sqrt{7+x}$

(iii) $y = \sqrt{3x+5}$

(d) (i) $y = \sqrt{(3-x)(x-5)}$ (ii) $y = \sqrt{(x-3)(x+5)}$

(iii) $y = \dfrac{1}{\sqrt{(3+x)(7+x)}}$ (iv) $y = \dfrac{1}{\sqrt{(x-3)(7+x)}}$

2. Find the range of the function, given its domain in each of the following cases.

(a) (i) $f(x) = 3x+10,\quad x \in \{1, 5, 7, -1, -2\}$

(ii) $f(x) = 2x^2+1,\quad x \in \{-3, 2, 4, 0\}$

(iii) $f(x) = x^2-x+2,\quad x \in \{1, 2, 3, 4, 5\}$

(b) (i) $f(x) = x - 2,\ 0 \le x \le 4$ (ii) $f(x) = 3x + 4,\ -1 \le x \le 2$

(c) (i) $f(x) = x^2,\ -5 \le x \le 5$ (ii) $f(x) = 2x,\ -3 \le x \le 3$

(iii) $f(x) = x^2 + 1,\ -2 \le x \le 2$ (iv) $f(x) = \sqrt{x},\ 0 \le x \le 25$

(d) (i) $f(x) = x + 5,\ x \in R$ (ii) $f(x) = 2x - 3,\ x \in R$

(iii) $f(x) = x^3,\ x \in R$ (iv) $f(x) = \dfrac{1}{x},\ \{x : x < 0\}$

(v) $f(x) = \dfrac{1}{x-2},\ \{x : x \le 1\}$ (vi) $f(x) = \dfrac{1}{3x-2},\ \{x : x \le 0\}$

(vii) $f(x) = \dfrac{2}{x},\ \{x : x > 0\}$ (viii) $f(x) = \dfrac{x}{x+5},\ \{x : x \ne -5\}$

15.15 CLASSIFICATION OF FUNCTIONS

Let f be a function from A to B. If every element of the set B is the image of at least one element of the set A i.e. if there is no unpaired element in the set B then we say that the function f maps the set A onto the set B. Otherwise we say that the function maps the set A into the set B.

Functions for which each element of the set A is mapped to a different element of the set B are said to be ***one-to-one.***

One-to-one function

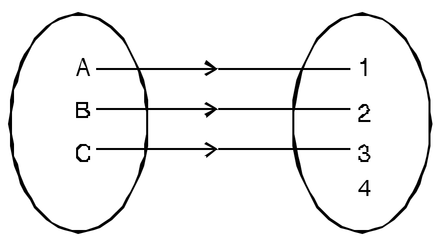


Fig.15.27

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 2, 3\}$

A function can map more than one element of the set A to the same element of the set B. Such a type of function is said to be ***many-to-one.***

Many-to-one function

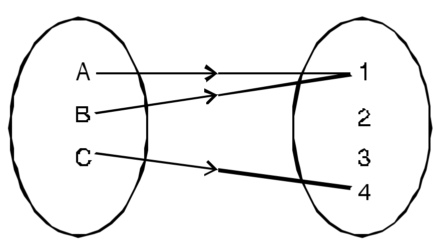


Fig. 15.28

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 4\}$

A function which is both one-to-one and onto is said to be a bijective function.

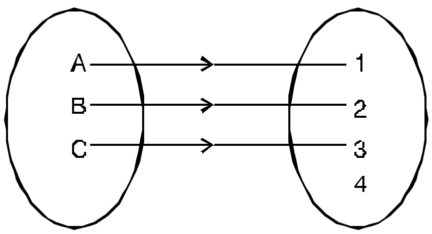


Fig. 15.29

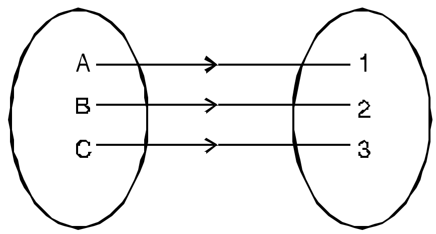


Fig. 15.30

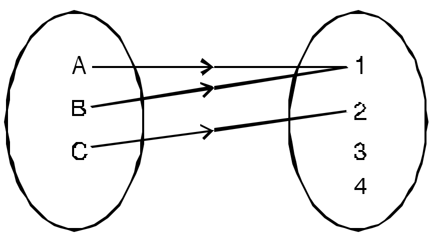


Fig. 15.31

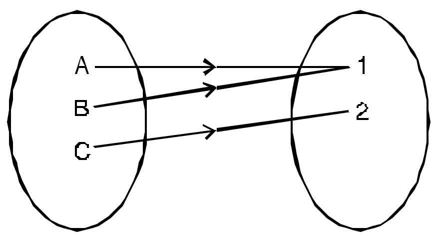


Fig. 15.32

Fig. 15.29 shows a one-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 15.30 shows a one-to-one function mapping $\{A, B, C\}$ onto $\{1, 2, 3\}$.

Fig. 15.31shows a many-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 15.32 shows a many-to-one function mapping $\{A, B, C\}$ onto $\{1, 2\}$.

Function shown in Fig. 15.30 is also a bijective Function.

Note : Relations which are one-to-many can occur, but they are not functions. The following figure illustrates this fact.

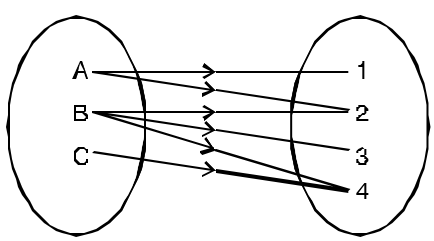


Fig. 15.33

Example 15.28 Without using graph prove that the function

$F : R \rightarrow R$ defiend by $f(x) = 4 + 3x$ is ***one-to-one***.

Solution : For a function to be one-one function

$$F(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall \quad x_1, x_2 \in \text{domain}$$

$\therefore$ Now $f(x_1) = f(x_2)$ gives

$4 + 3x_1 = 4 + 3x_2$ or $x_1 = x_2$

$\therefore$ F is a ***one-one function***.

Example 15.29 Prove that

$F : R \to R$ defined by $f(x) = 4x^3 - 5$ is a bijection

Solution : Now $f(x_1) = f(x_2) \quad \forall\ x_1, x_2$ Domain

$\therefore \quad 4x_1^3 - 5 = 4x_2^3 - 5$

$\Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad x_1^3 - x_2^3 = 0 \Rightarrow \quad (x_2 - x_1)\left(x_1^2 + x_1x_2 + x_2^2\right) = 0$

$\Rightarrow \quad x_1 = x_2$ or

$x_1^2 + x_1x_2 + x_2^2 = 0$ (rejected). It has no real value of x_1 and x_2.

$\therefore$ F is a ***one-one function***.

Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.

We have $y = 4x^3 - 5$ or $x = \left(\dfrac{y+5}{4}\right)^{1/3}$

$\therefore$ For each $y \in$ codomain $\exists\ x \in$ domain such that $f(x) = y$.

Thus F is ***onto function***.

$\therefore$ F is a bijection.

Example 15.30 Prove that $F : R \to R$ defined by $F(x) = x^2 + 3$ is neither ***one-one*** nor ***onto function***.

Solution : We have $F(x_1) = F(x_2) \quad \forall\ x_1, x_2 \in$ domain giving

$x_1^2 + 3 = x_2^2 + 3 \Rightarrow x_1^2 = x_2^2$

or $\quad x_1^2 - x_2^2 = 0 \Rightarrow x_1 = x_2$ or $x_1 = -x_2$

or $\quad$ F is not ***one-one function***.

Again let $y = F(x)$ where $y \in$ codomain

$x \in$ domain.

$\Rightarrow \quad y = x^2 + 3 \quad \Rightarrow \quad x = \pm\sqrt{y - 3}$

$\Rightarrow \quad \forall\ y < 3$ no real value of x in the domain.

$\therefore$ F is not an ***onto finction***.

15.16 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

$$y = x^2$$

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

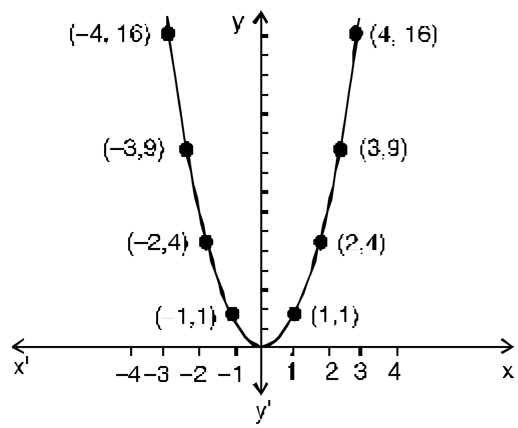


Fig. 15.34

Does this represent a function?

Yes, this represent a function because corresponding to each value of $x\ \exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

$$x^2 + y^2 = 25$$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

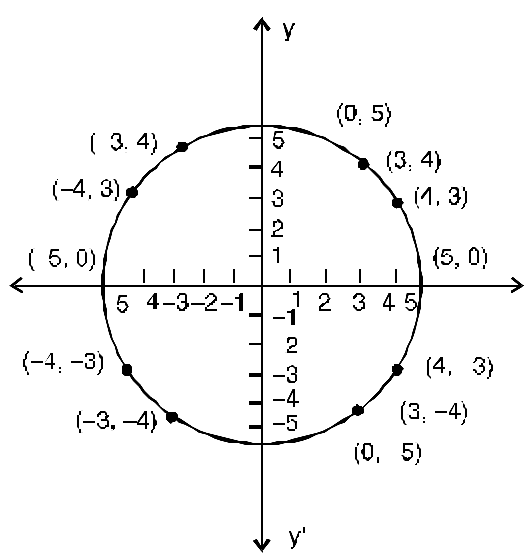


Fig. 15.35

This graph represents a circle.

Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

CHECK YOUR PROGRESS 15.7

1. (i) Does the graph represent a function?

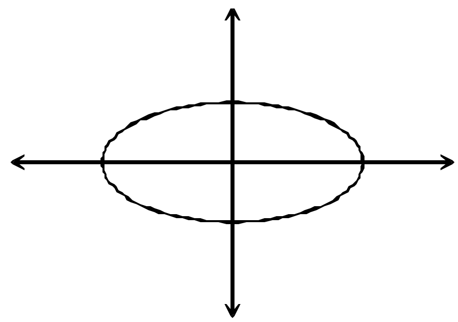


Fig. 15.36

(ii) Does the graph represent a function ?

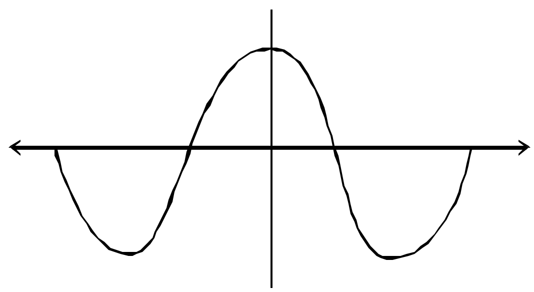


Fig. 15.37

2. Which of the following functions are into function ?

(a)

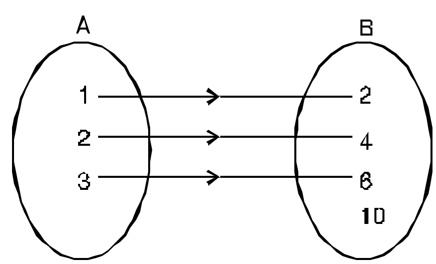


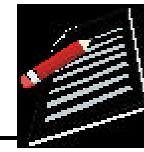
Fig.15.38

(b) $f : N \to N,$ defined as $f(x) = x^2$

Here N represents the set of natural numbers.

(c) $f : N \to N,$ defined as $f(x) = x$

3. Which of the following functions are onto function if $f : R \to R$



(a) $f(x) = 115x + 49$ (b) $f(x) = |x|$

4. Which of the following functions are one-to-one functions ?

(a) $f : \{20,21,22\} \to \{40,42,44\}$ defined as $f(x) = 2x$

(b) $f : \{7,8,9\} \to \{10\}$ defined as $f(x) = 10$

(c) $f : I \to R$ defined as $f(x) = x^3$

(d) $f : R \to R$ defined as $f(x) = 2 + x^4$

(d) $f : N \to N$ defined as $f(x) = x^2 + 2x$

5. Which of the following functions are many-to-one functions ?

(a) $f : \{-2,-1,1,2\} \to \{2,5\}$ defined as $f(x) = x^2 + 1$

(b) $f : \{0,1,2\} \to \{1\}$ defined as $f(x) = 1$

(c)

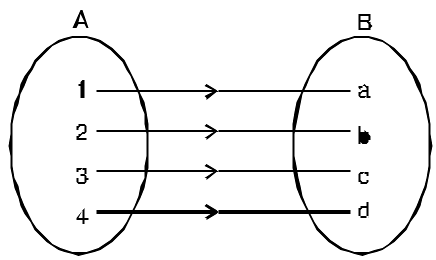


Fig.15.39

(d) $f : N \to N$ defined as $f(x) = 5x + 7$

6. Draw the graph of each of the following functions :

(a) $y = 3x^2$ (b) $y = -x^2$ (c) $y = x^2 - 2$

(d) $y = 5 - x^2$ (e) $y = 2x^2 + 1$ (f) $y = 1 - 2x^2$

7. Which of the following graphs represents a function ?

(a)

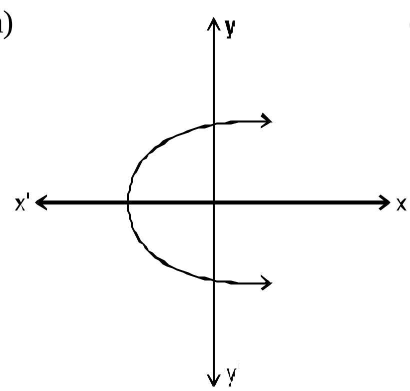


Fig. 15.40

(b)

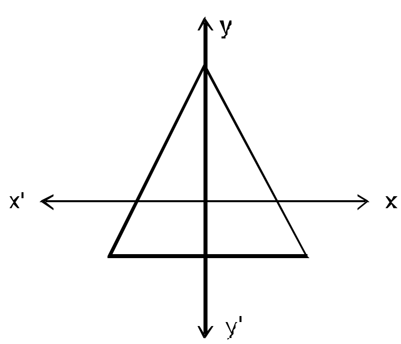


Fig. 15.41

(c)

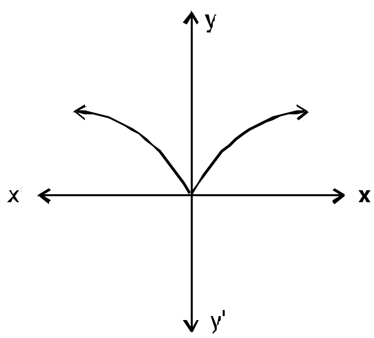


Fig. 15.42

(d)

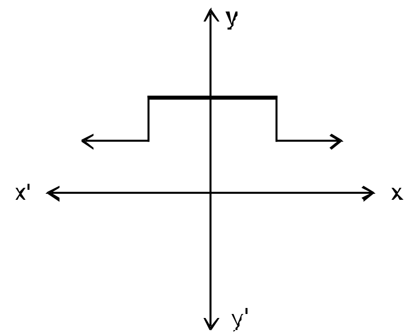


Fig. 15.43

(e)

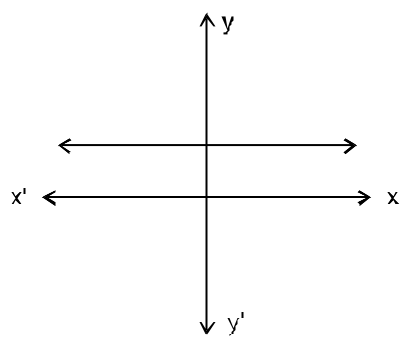


Fig .15.44

Hint : If any line || to y-axis cuts the graph in more than one point, graph does not represent a function.

15.17 SOME SPECIAL FUNCTIONS

15.17.1 Monotonic Function

Let $F : A \to B$ be a function then F is said to be monotonic on an interval (a,b) if it is either increasing or decreasing on that interval.

For function to be increasing on an interval (a,b)

$$x_1 < x_2 \quad \Rightarrow F(x_1) < F(x_2) \quad \forall\, x_1\ x_2 \in (a, b)$$

and for function to be decreasing on (a,b)

$$x_1 < x_2 \quad \Rightarrow F(x_1) > F(x_2) \quad \forall\, x_1\ x_2 \in (a, b)$$

A function may not be monotonic on the whole domain but it can be on different intervals of the domain.

Consider the function $F : R \to R$ defined by $f(x) = x^2$.

Now $\forall \ x_1, x_2 \in [0, \infty]$

$$x_1 < x_2 \quad \Rightarrow F(x_1) < F(x_2)$$

$\Rightarrow$ F is a ***Monotonic Function*** on $[0, \infty]$.

($\because$ It is only increasing function on this interval)

But $\forall\ x_1, x_2 \in (-\infty, 0)$

$x_1 < x_2 \Rightarrow F(x_1) > F(x_2)$

$\Rightarrow$ F is a ***Monotonic Function*** on $[-\infty, 0]$

($\because$ It is only a decreasing function on this interval)

Therefore if we talk of the whole domain given function is not monotonic on R but it is monotonic on $(-\infty, 0)$ and $(0, \infty)$.

Again consider the function $F : R \to R$ defined by $f(x) = x^3$.

Clearly $\forall\ x_1\ x_2 \in$ domain

$x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

$\therefore$ Given function is ***monotonic*** on R i.e. on the whole domain.

15.17.2 Even Function

A function is said to be an even function if for each x of domain

$F(-x) = F(x)$

For example, each of the following is an ***even function***.

(i) If $F(x) = x^2$ then $F(-x) = (-x)^2 = x^2 = F(x)$

(ii) If $F(x) = \cos x$ then $F(-x) = \cos(-x) = \cos x = F(x)$

(iii) If $F(x) = |x|$ then $F(-x) = |-x| = |x| = F(x)$

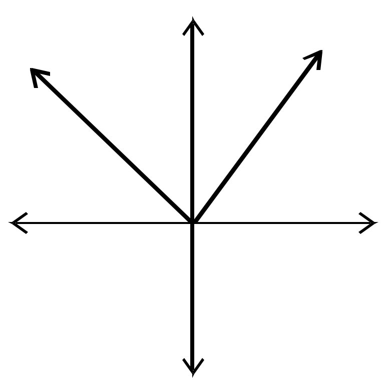


Fig. 15.45

The graph of this even function (modulus function) is shown in the figure above.

Observation

Graph is symmetrical about y-axis.

15.17.3 Odd Function

A function is said to be an odd function if for each x

$$f(-x) = -f(x)$$

For example,

(i) If $f(x) = x^3$

then $f(-x) = (-x)^3 = -x^3 = -f(x)$

(ii) If $f(x) = \sin x$

then $f(-x) = \sin(-x) = -\sin x = -f(x)$

Graph of the odd function y = x is given in Fig.15.46

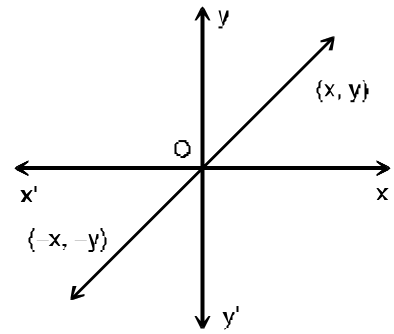


Fig. 15.46

Observation

Graph is symmetrical about origin.

15.17.4 Greatest Integer Function (Step Function)

$f(x) = [x]$ which is the greatest integer less than or equal to x.

$f(x)$ is called Greatest Integer Function or Step Function. Its graph is in the form of steps, as shown in Fig. 16.47.

Let us draw the graph of $y = [x], x \in R$

$[x] = 1, \quad 1 \le x < 2$

$[x] = 2, \quad 2 \le x < 3$

$[x] = 3, \quad 3 \le x < 4$

$[x] = 0, \quad 0 \le x < 1$

$[x] = -1, \quad -1 \le x < -0$

$[x] = -2, \quad -2 \le x < -1$

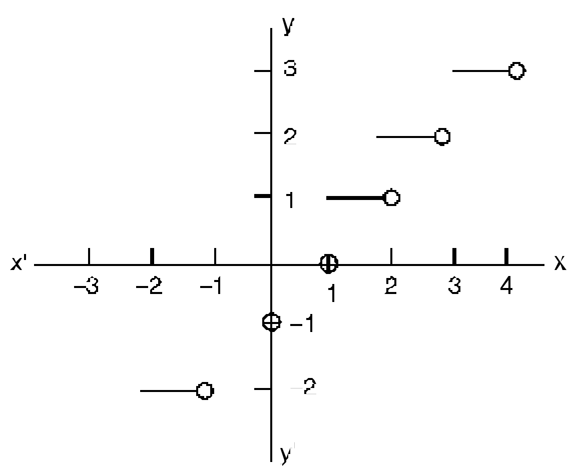


Fig. 15.47

- Domain of the step function is the set of real numbers.
- Range of the step function is the set of integers.

15.17.5 Polynomial Function

Any function defined in the form of a polynomial is called a polynomial function.

For example,

(i) $f(x) = 3x^2 - 4x - 2$

(ii) $f(x) = x^3 - 5x^2 - x + 5$

(iii) $\quad f(x) = 3$

are all polynomial functions.

Note : Functions of the type $f(x) = k$, where k is a constant is also called a constant function.

15.17.6 Rational Function

Function of the type $f(x) = \dfrac{g(x)}{h(x)}$, where $h(x) \neq 0$ and $g(x)$ and $h(x)$ are polynomial functions are called rational functions.

For example, $\quad f(x) = \dfrac{x^2 - 4}{x + 1}, x \neq -1$

is a rational function.

15.17.7 Reciprocal Function

Functions of the type $y = \dfrac{1}{x}, x \neq 0$ is called a reciprocal function.

15.17.8 Exponential Functions

A swiss mathematician Leonhard Euler introduced a number e in the form of an infinite series. In fact

$$e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots + \frac{1}{\underline{|n}} + \ldots \qquad \ldots\ldots(1)$$

It is well known that the sum of its infinite series tends to a finite limit (i.e., this series is convergent) and hence it is a positive real number denoted by e. This number e is a transcendental irrational number and its value lies between 2 an 3.

Consider now the infinite series

$$1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^n}{\underline{|n}} + \ldots$$

It can be shown that the sum of its infinite series also tends to a finite limit, which we denote by e^x. Thus,

$$e^x = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots + \frac{x^n}{\underline{|n}} + \ldots \qquad \ldots\ldots(2)$$

This is called the **Exponential Theorem** and the infinite series is called the **exponential series**. We easily see that we would get (1) by putting x = 1 in (2).

The function $f(x) = e^x$, where x is any real number is called an **Exponential Function**.

The graph of the exponential function

$$y = e^x$$

is obtained by considering the following important facts :

(i) As x increases, the y values are increasing very rapidly, whereas as x decreases, the y values are getting closer and closer to zero.

(ii) There is no x-intercept, since $e^x \neq 0$ for any value of x.

(iii) The y intercept is 1, since $e^0 = 1$ and $e \neq 0$.

(iv) The specific points given in the table will serve as guidelines to sketch the graph of e^x (Fig. 15.48).

x	−3	−2	−1	0	1	2	3
$y = e^x$	0.04	0.13	0.36	1.00	2.71	7.38	20.08

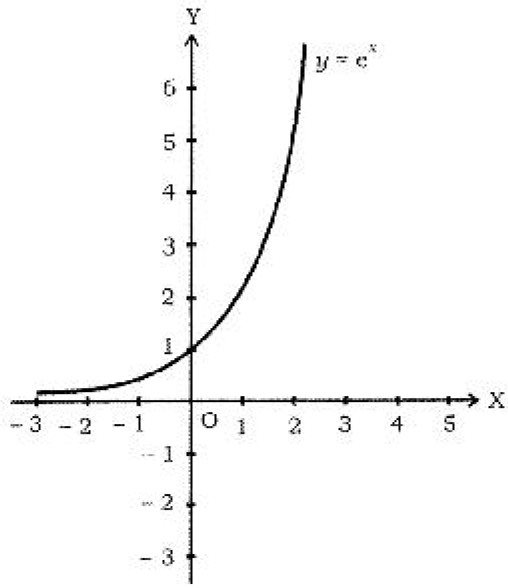


Fig. 15.48

If we take the base different from e, say a, we would get exponential function

$$f(x) = a^x \text{ ,provbided } a > 0, \quad a \neq 1.$$

For example, we may take a = 2 or a = 3 and get the graphs of the functions

$$y = 2^x \text{ (See Fig. 15.49)}$$

and $$y = 3^x \text{(See Fig. 15.50)}$$

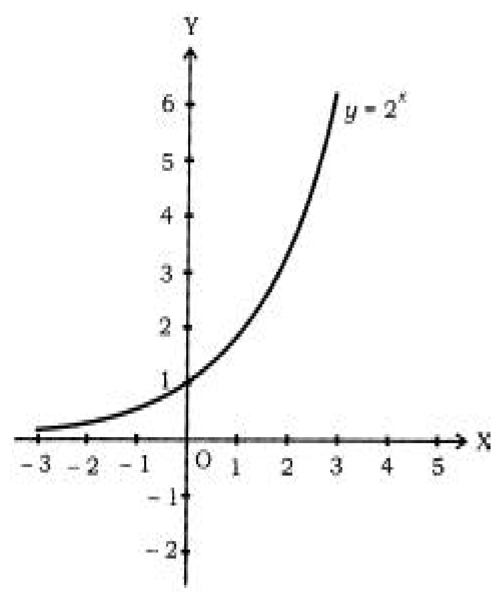


Fig. 15.49

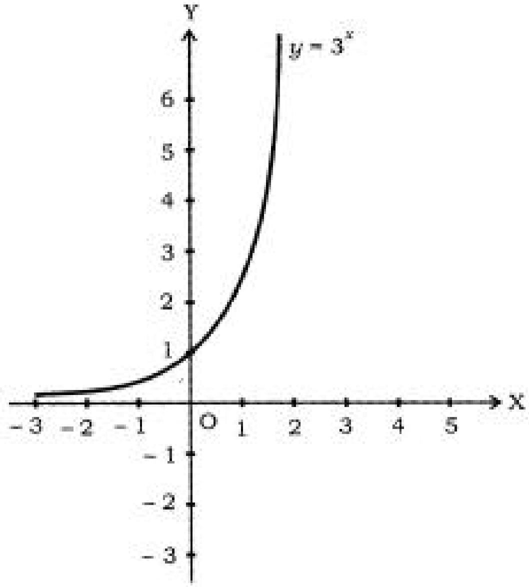


Fig. 15.50

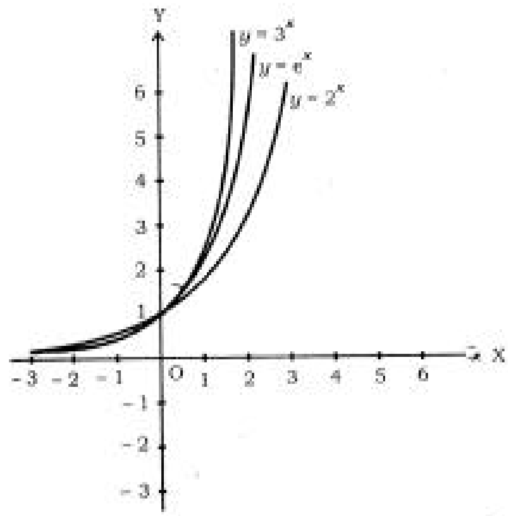


Fig. 15.51

15.17.9 Logarithmic Functions

Consider now the function

$$y = e^x \qquad .....(3)$$

We write it equivalently as

$$x = \log_e y$$

Thus, $\quad y = \log_e x \qquad .....(4)$

is the inverse function of $y = e^x$

The base of the logarithm is not written if it is e and so $\log_e x$ is usually written as log x.

As $y = e^x$ and $y = \log x$ are inverse functions, their graphs are also symmetric w.r.t. the line $y = x$

The graph of the function $y = \log x$ can be obtained from that of $y = e^x$ by reflecting it in the line y = x.

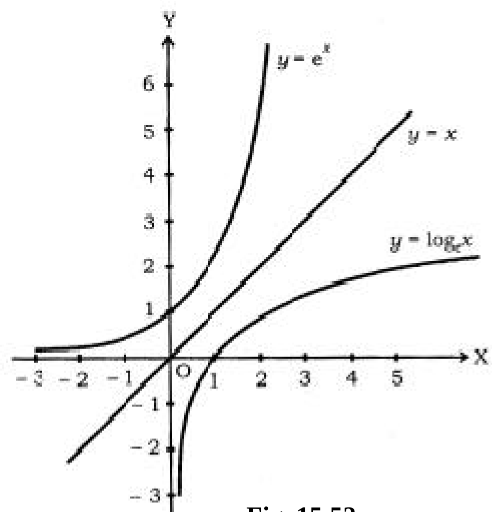


Fig. 15.52

Note

(i) The learner may recall the laws of indices which you have already studied in the Secondary Mathematics :

If a > 0, and m and n are any rational numbers, then

$$a^m \cdot a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^0 = 1$$

(ii) The corresponding laws of logarithms are

$$\log_a(mn) = \log_a m + \log_a n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a(m^n) = n \log_a m$$

$$\log_b m = \frac{\log_a m}{\log_a b}$$

or $\log_b m = \log_a m \log_b a$

Here a, b > 0, $a \neq 1$, $b \neq 1$.

CHECK YOUR PROGRESS 15.8

1. Tick mark the correct statement.

   (i) Function $f(x) = 2x^4 + 7x^2 + 9x$ is an even function.

   (ii) Odd function is symmetrical about y-axis.

   (iii) $f(x) = x^{1/2} - x^3 + x^5$ is a polynomial function.

   (iv) $f(x) = \frac{x-3}{3+x}$ is a rational function for all $x \in R$.

   (v) $f(x) = \frac{\sqrt{5}}{3}$ is a constant function.

   (vi) $f(x) = \frac{1}{x}$

   Domain of the function is the set of real numbers except 0.

   (vii) Greatest integer function is neither even nor odd.

2. Which of the following functions are even or odd functions ?

   (a) $f(x) = \frac{x^2-1}{x+1}$ (b) $f(x) = \frac{x^2}{5+x^2}$ (c) $f(x) = \frac{1}{x^2+5}$

   (d) $f(x) = \frac{2}{x^3}$ (e) $f(x) = \frac{x}{x^2+1}$ (f) $f(x) = \frac{5}{x-5}$

(g) $f(x) = \dfrac{x-3}{3+x}$ (h) $f(x) = x - x^3$

3. Draw the graph of the function $y = [x] - 2$.

4. Specify the following functions as polynomial function, rational function, reciprocal function or constant function.

(a) $y = 3x^8 - 5x^7 + 8x^5$ (b) $y = \dfrac{x^2 + 2x}{x^3 - 2x + 3}, \; x^3 - 2x + 3 \neq 0$

(c) $y = \dfrac{3}{x^2}, \; x \neq 0$ (d) $y = 3 + \dfrac{2x+1}{x}, \; x \neq 0$

(e) $y = 1 - \dfrac{1}{x}, \; x \neq 0$ (f) $y = \dfrac{x^2 - 5x + 6}{x - 2}, \; x \neq 2$

(g) $y = \dfrac{1}{9}$.

15.18 COMPOSITION OF FUNCTIONS

Consider the two functions given below:

$$y = 2x + 1, \quad x \in \{1,2,3\}$$

$$z = y + 1, \quad y \in \{3,5,7\}$$

Then *z* is the composition of two functions *x* and *y* because *z* is defined in terms of *y* and *y* in terms of *x*.

Graphically one can represent this as given below :

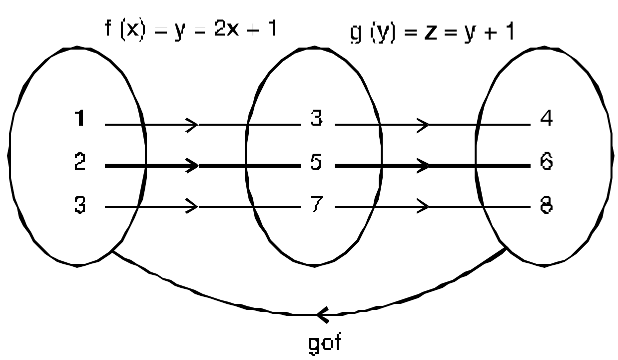


Fig. 15.53

The composition, say, gof of function g and f is defined as function g of function f.

If $f : A \to B$ and $g : B \to C$

then $g \circ f : A$ to C

Let $f(x) = 3x + 1$ and $g(x) = x^2 + 2$

Then $fog(x) = f(g(x))$

$$= f\left(x^2+2\right)$$

$$= 3\left(x^2+2\right)+1 = 3x^2+7 \qquad \text{(i)}$$

and $\quad (gof)(x) = g(f(x))$

$$= g(3x+1)$$

$$= (3x+1)^2+2 = 9x^2+6x+3 \qquad \text{(ii)}$$

Check from (i) and (ii), if

$$fog = gof$$

Evidently, $\quad fog \neq gof$

Similarly, $(fof)(x) = f(f(x)) = f(3x+1)$ [Read as function of function f].

$$= 3(3x+1)+1$$

$$= 9x+3+1 = 9x+4$$

$(gog)(x) = g(g(x)) = g\left(x^2+2\right)$ [Read as function of function g]

$$= \left(x^2+2\right)^2+2$$

$$= x^4+4x^2+4+2$$

$$= x^4+4x^2+6$$

Example 15.31 If $f(x) = \sqrt{x+1}$ and $g(x) = x^2+2$, calculate *fog* and *gof*.

Solution : $\quad fog(x) = f(g(x))$

$$= f\left(x^2+2\right)$$

$$= \sqrt{x^2+2+1}$$

$$= \sqrt{x^2+3}$$

$$(gof)(x) = g(f(x))$$

$$= g\left(\sqrt{x+1}\right)$$

$$= \left(\sqrt{x+1}\right)^2+2$$

$$= x+1+2$$

$$= x+3.$$

Here again, we see that $(fog) \neq gof$

Example 15.32 If $f(x) = x^3$, $f : R \to R$

$$g(x) = \frac{1}{x}, \quad g : R-\{0\} \to R-\{0\}$$

Find *fog* and *gof*.

Solution : $(fog)(x) = f(g(x))$

$$= f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$$

$$(gof)(x) = g(f(x))$$

$$= g(x^3) = \frac{1}{x^3}$$

Here we see that $fog = gof$

Note : We observe from Example 15.31 and Example 15.32 that *fog* and *gof* may or may not be equal.

CHECK YOUR PROGRESS 15.9

1. Find *fog, gof, fof* and *gog* for the following functions :

   $f(x) = x^2 + 2, \quad g(x) = 1 - \frac{1}{1-x}, \; x \neq 1.$

2. For each of the following functions write *fog, gof, fof* and *gog*.

   (a) $f(x) = x^2 - 4, \; g(x) = 2x + 5$

   (b) $f(x) = x^2, \; g(x) = 3$

   (c) $f(x) = 3x - 7, \; g(x) = \frac{2}{x}, \; x \neq 0$

3. Let $f(x) = |x|, \; g(x) = [x]$. Verify that $fog \neq gof$.

4. Let $f(x) = x^2 + 3, \; g(x) = x - 2$

   Prove that $fog \neq gof$ and $f\left(f\left(\frac{3}{2}\right)\right) = g\left(f\left(\frac{3}{2}\right)\right)$

5. If $f(x) = x^2, \; g(x) = \sqrt{x}$. Show that $fog = gof$.

6. Let $f(x) = |x|, \; g(x) = (x)^{\frac{1}{3}}, \; h(x) = \frac{1}{x}; \; x \neq 0.$

   Find (a) *fog* (b) *goh* (c) *foh* (d) *hog* (e) *fogoh*

   $\left\{ \textbf{Hint :} \; (fogoh)(x) = f(g(h(x))) = f\left(g\left(\frac{1}{x}\right)\right) \right\}$

MODULE - IV
Functions

Notes

15.19 INVERSE OF A FUNCTION

(A) Consider the relation

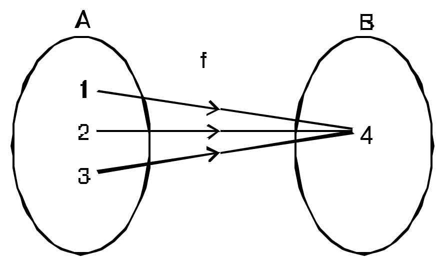


Fig. 15.54

This is a many-to-one function. Now let us find the inverse of this relation. Pictorially, it can be represented as

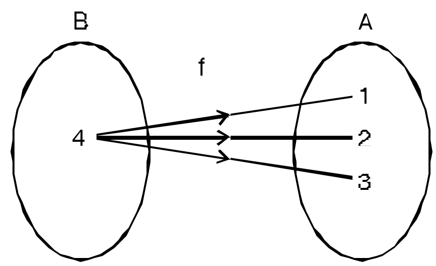


Fig 15.55

Clearly this relation does not represent a function. (Why ?)

(B) Now take another relation

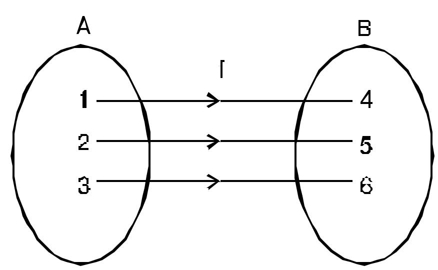


Fig.15.56

It represents one-to-one onto function. Now let us find the inverse of this relation, which is represented pictorially as

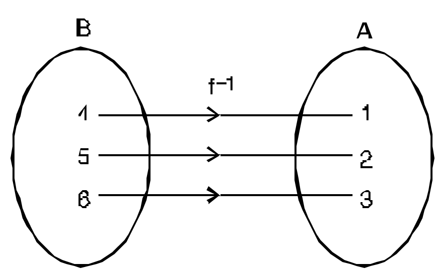


Fig. 15.57

This represents a function.

(C) Consider the relation

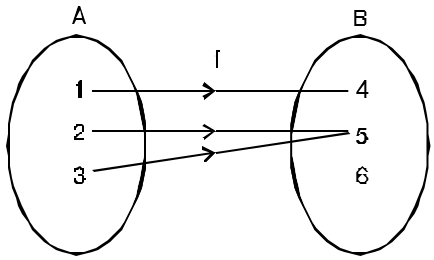


Fig. 15.58

Ir represents many-to-one function. Now find the inverse of the relation.

Pictorially it is represented as

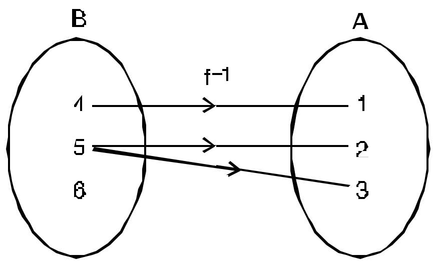


Fig. 15.59

This does not represent a function, because element 6 of set B is not associated with any element of A. Also note that the elements of B does not have a unique image.

(D) Let us take the following relation

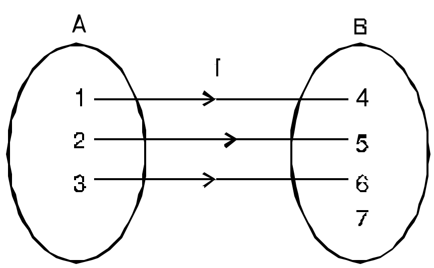


Fig. 15.60

It represent one-to-one into function.

Find the inverse of the relation.

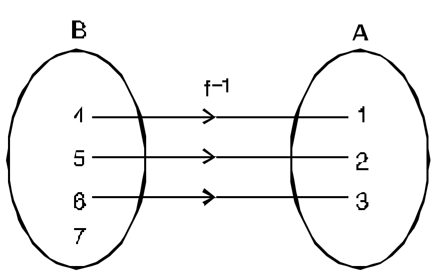


Fig. 15.61

It does not represent a function because the element 7 of *B* is not associated with any element of *A*. From the above relations we see that we may or may not get a relation as a function when we find the inverse of a relation (function).

We see that the inverse of a function exists only if the function is one-to-one onto function i.e. only if it is a bijective function.

CHECK YOUR PROGRESS 15.10

1 (i) Show that the inverse of the function

$$y = 4x - 7 \text{ exists.}$$

(ii) Let f be a one-to-one and onto function with domain A and range B. Write the domain and range of its inverse function.

2. Find the inverse of each of the following functions (if it exists) :

(a) $f(x) = x + 3 \quad \forall x \in R$

(b) $f(x) = 1 - 3x \quad \forall x \in R$

(c) $f(x) = x^2 \quad \forall x \in R$

(d) $f(x) = \dfrac{x+1}{x}, \quad x \neq 0 \quad x \in R$

LET US SUM UP

- Set is a well defined collection of objects.
- To represent a set in Roster form all elements are to be written but in set builder form a set is represented by the common property.
- If the elements of a set can be counted then it is called a finite set and if the elements cannot be counted, it is infinite.
- If each element of set *A* is an element of set *B* also then *A* is called sub set of *B*.
- For two sets *A* and *B*, $A - B$ is a set of those elements which are in *A* but not in *B*.
- Complement of a set *A* is a set of those elements which are in the universal set but not in *A*. i.e. $A^c = U - A$
- Intersection of two sets is a set of those elements when belong to both the sets.
- Union of two sets is a set of those elements which belong to either of the two sets.
- Cartesian product of two sets *A* and *B* is the set of all ordered pairs of the elements of *A* and *B*. It is denoted by $A \times B$. i.e.

  $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.
- Relation is a sub set of A×B where *A* and *B* are sets.

  i.e. $R \subseteq A \times B = \{(a, b) : a \in A \text{ and } b \in B \text{ and } aRb\}$